(book 216)

ex: Find the length of the curve

$x = t^3 \qquad y = \dfrac{3t^2}{2}, \qquad 0 \le t \le \sqrt{3}$

$$\text{length} = \int_0^{\sqrt{3}} \sqrt{\underbrace{(3t^2)^2}_{\left(\frac{dx}{dt}\right)^2} + \underbrace{(3t)^2}_{\left(\frac{dy}{dt}\right)^2}}\; dt$$

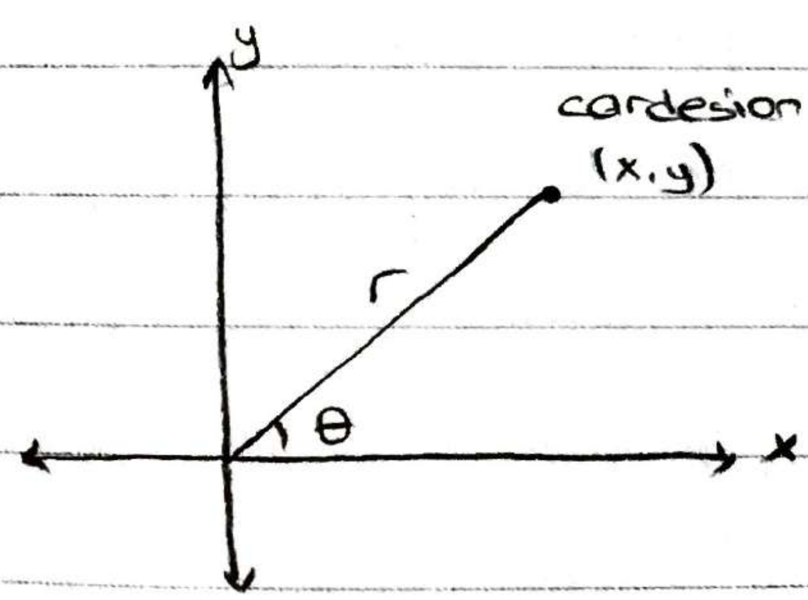
$$= 3\int_0^{\sqrt{3}} \sqrt{t^4+t^2}\; dt$$

$$= 3 \cdot \int_0^{\sqrt{3}} t\sqrt{t^2+1}\; dt$$

$t^2 + 1 = u \qquad 2t\,dt = du$

$$= 3\int_1^4 \sqrt{u}\,\frac{du}{2} = \frac{3}{2} \cdot \left.\frac{u^{3/2}}{3/2}\right|_1^4 = 4^{3/2} - 1 = 7$$

11.3 POLAR COORDINATES



cardesion coords (x,y) $\xrightarrow[\text{polar coord.}]{}$ (r, θ)

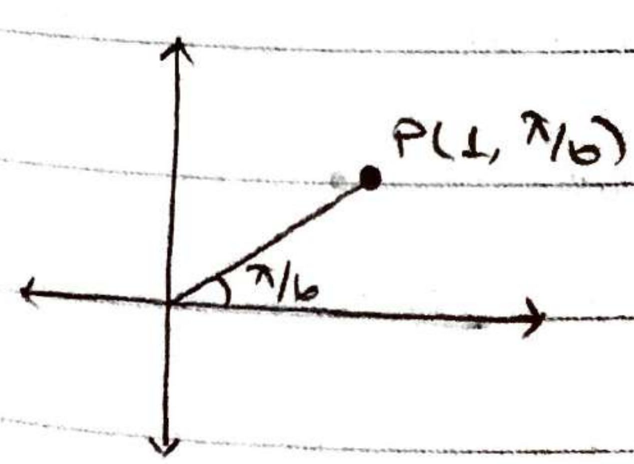
r ↓ directed distance

θ ↑ directed angle

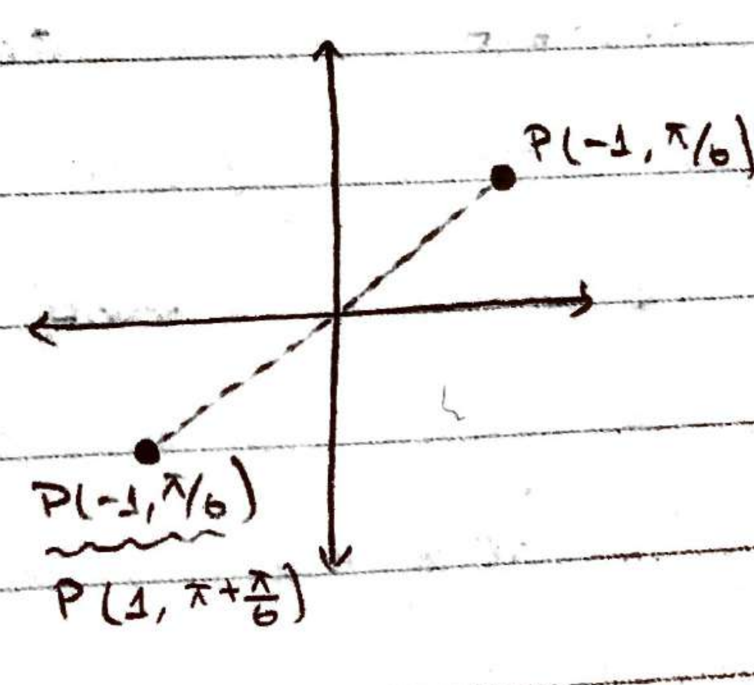
$-\infty < r < +\infty$

$-\infty < \theta < +\infty$

$P(1, \pi/6)$



$P(-1, \pi/6)$

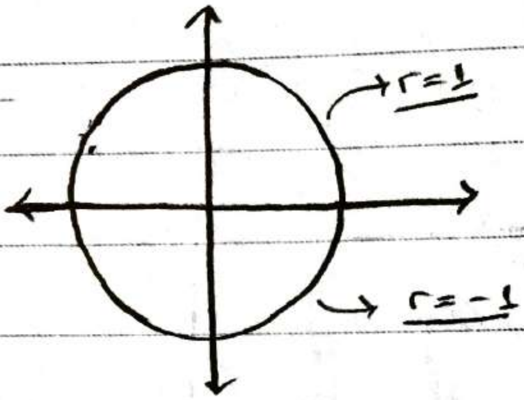


$P(-r,\theta) = P(r, \pi+\theta)$

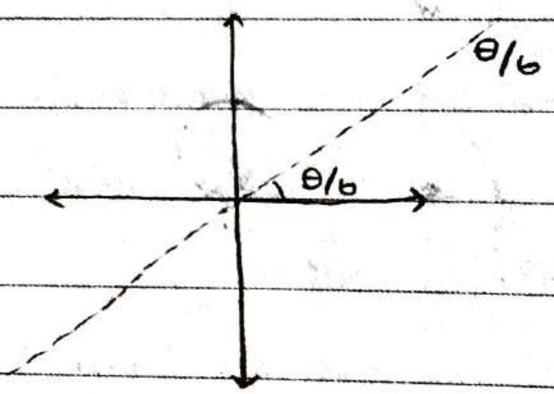
π radians = $180°$

Polar coordinates are not unique.

ex: $r=1$ in polar coordinates.



ex: $\theta = \pi/6$ $-\infty < r < +\infty$



$x = r\cos\theta$

$y = r\sin\theta$

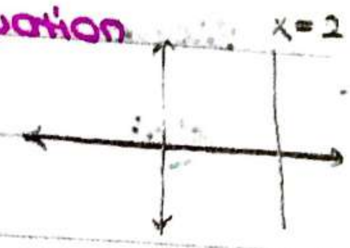
$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

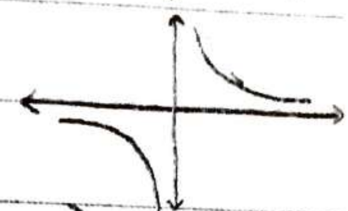
$r^2 = x^2+y^2$

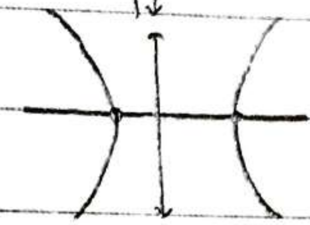
$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$

$\tan\theta = y/x$

polar equation	cartesian equation
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$







$r = 1 + 2r\cos\theta \Rightarrow r^2 = (1+2x)^2$ $\qquad x^2+y^2 = (1+2x)^2$

ex: Find a polar equation for the circle

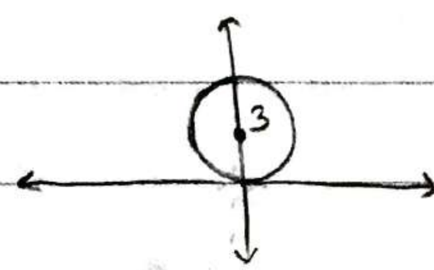
$x^2 + (y-3)^2 = 9$

$\underbrace{x^2 + y^2} - 6y + \cancel{9} = \cancel{9}$

$r^2 - 6r\sin\theta = 0$

$r = 6\sin\theta$ or $r = 0$

$r = 6\sin\theta$ includes both $\theta = 0 \Rightarrow r = 0$



11.4 GRAPHING POLAR COORDINATE EQUATIONS

Symetry C = graph of a curve

1) If C is symetric with respect to x-axis and if (r,θ) is on $C \Rightarrow (r,-\theta)$ is on C.

2) If C is symetric about the y-axis and if (r,θ) is on C $(r,\pi-\theta)$ is on C

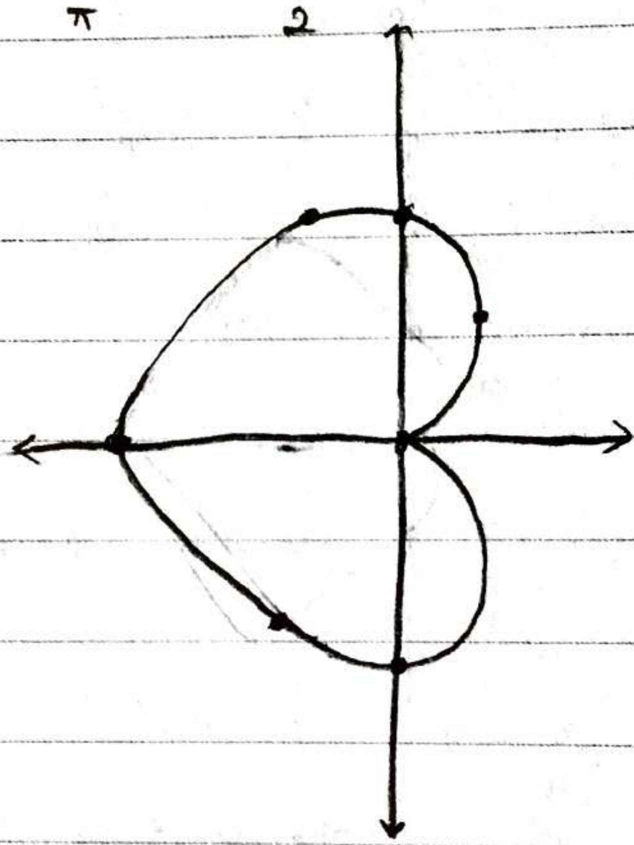
3) If C is symetric about origin and if (r,θ) on C $(-r,\theta)$ is on C. Also $(r,\pi+\theta)$

Polar curve. $r = f(\theta)$

ex: Graph $r = 1 - \cos\theta$ on the cartesian plane.

θ	r
0	0
$\pi/4$	$1 - \frac{\sqrt{2}}{2}$
$\pi/2$	1
$3\pi/4$	$1 + \sqrt{2}/2$
π	2

θ	r
$5\pi/4$	$1 + \sqrt{2}/2$
$3\pi/2$	1
$7\pi/4$	$1 - \sqrt{2}/2$
2π	0

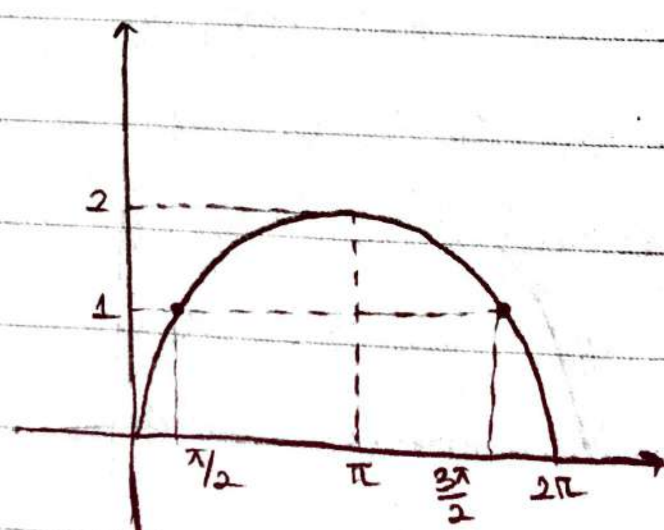


if (r, θ) satisfies $r = 1 - \cos\theta$

$(r, -\theta) \Rightarrow r = 1 - \cos(-\theta) = 1 - \cos\theta$

This is why graph is symetric with respect to x-axis.

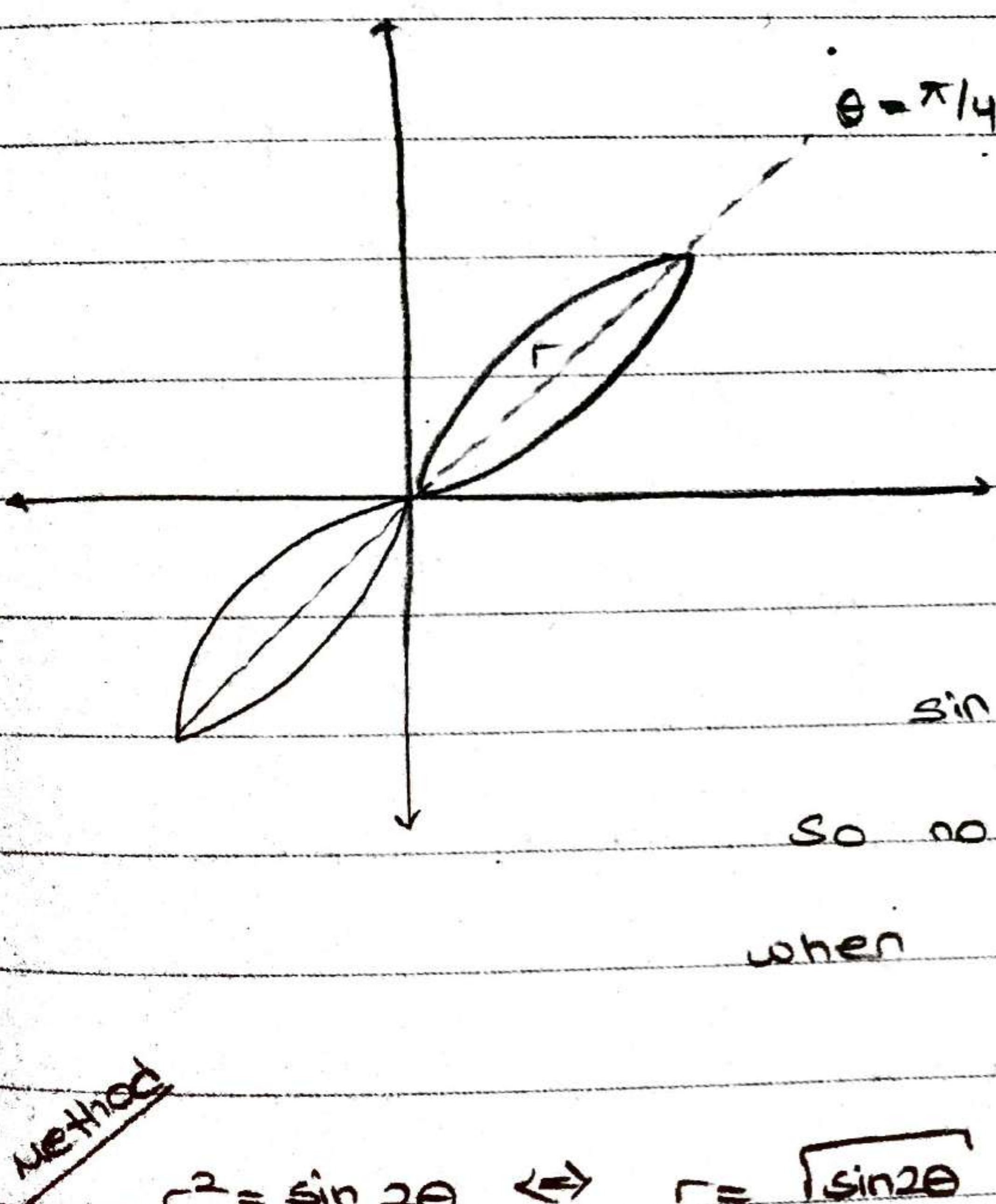
2nd Method



ex: Graph the curve $r^2 = \sin 2\theta$ in the cartesian xy-plane.

θ	r	
0	0	
$\pi/4$	∓ 1	
$\pi/2$	0	
$3\pi/4$		$r^2 = -1$ no solution
π	0	

$\sin 2\theta < 0$ in this interval



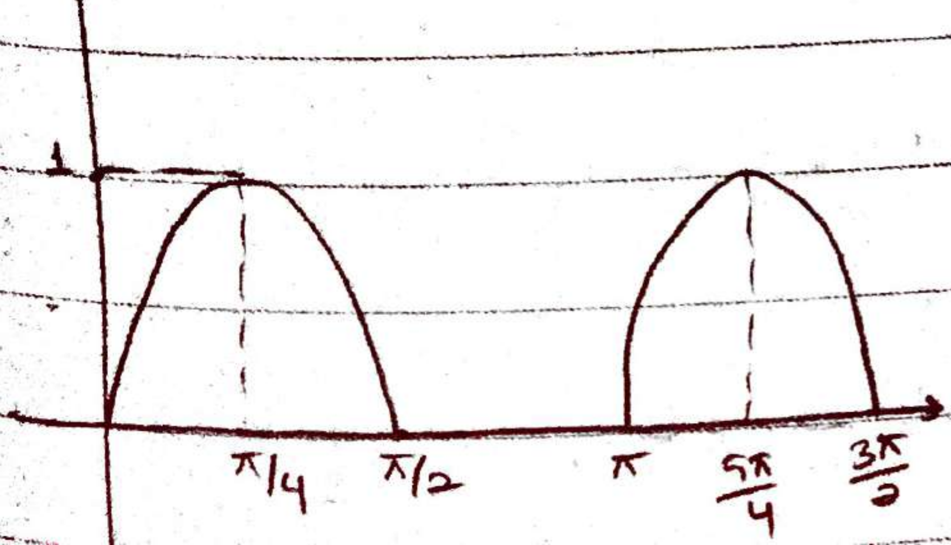
$\sin 2\theta < 0$ if $\frac{\pi}{2} < \theta < \pi$

So no solution for $r = \sin 2\theta$

when $\frac{\pi}{2} < \theta < \pi$

2nd method

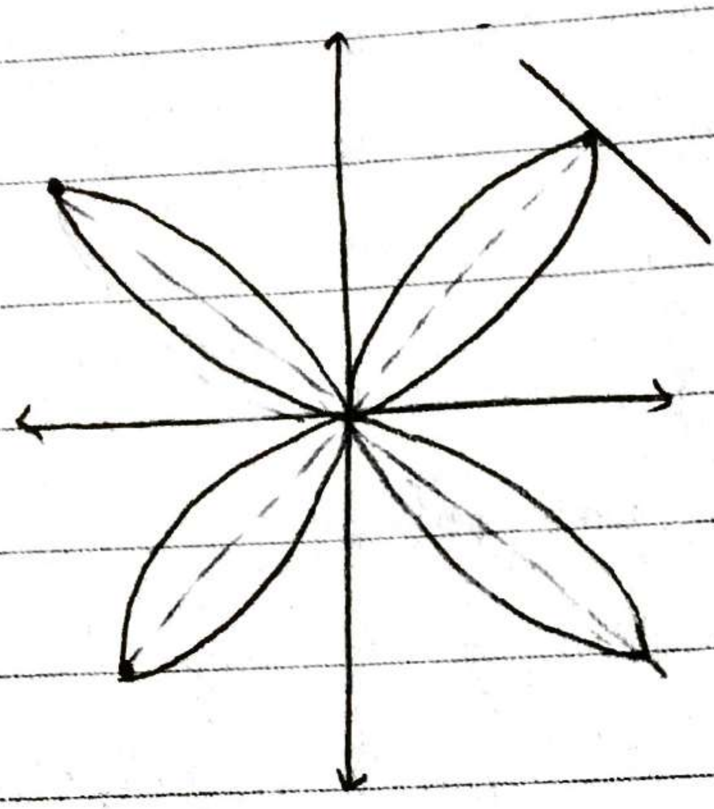
$r^2 = \sin 2\theta \iff r = \sqrt{\sin 2\theta}$ or $r = -\sqrt{\sin 2\theta}$



(book 19)

ex: Find the slope of the curve $r = \sin 2\theta$

at $\theta = \pi/4$ $\quad \frac{dy}{dx} = ?$



$\theta = \frac{\pi}{4} \Rightarrow r = 1$

$x = r\cos\theta = \sqrt{2}/2$

$y = r\sin\theta = \sqrt{2}/2$

$r^2 = \sin^2 2\theta = (2\sin\theta\cos\theta)^2$

$= 4\sin^2\theta\cos^2\theta$

$r^6 = (x^2+y^2)^3 = 4y^2x^2$

Find $\frac{dy}{dx}$. Implict differention

$3(x^2+y^2)^2\left(2x + 2y\frac{dy}{dx}\right)$

$= 4 \cdot 2y\frac{dy}{dx}x^2 + 4y^2 2x$

$x = \frac{\sqrt{2}}{2} = y$

$3 \cdot 2\frac{\sqrt{\cancel{2}}}{\cancel{2}}\left(1 + \frac{dy}{dx}\right) = 8 \cdot \frac{\sqrt{\cancel{2}}}{\cancel{2}}\left(\frac{1}{2}\frac{dy}{dx} + \frac{1}{2}\right)$

$6\left(1 + \frac{dy}{dx}\right) = 4\left(1 + \frac{dy}{dx}\right)$

$\Rightarrow 2\left(1 + \frac{dy}{dx}\right) = 0 \rightsquigarrow \frac{dy}{dx} = -1$

Recall: $\left.\begin{array}{l} x = f(t) \\ y = g(t) \end{array}\right\}$ parametric curve

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

Today: $r = f(\theta)$ polar curve

$\left.\begin{array}{l} x = r\cos\theta = f(\theta).\cos\theta \\ y = r\sin\theta = f(\theta).\sin\theta \end{array}\right\}$ $\theta \to$ parameter.

$$\frac{dy}{dx} = \frac{dy/dx}{dx/d\theta} = \frac{f(\theta).\sin\theta + f'(\theta)\cos\theta}{f''(\theta)\cos\theta - f(\theta)\sin\theta}$$

ex! Let's problem revisited!

$$r = f(\theta) = \sin 2\theta$$

$$\frac{dy}{dx} = \frac{2\cos 2\theta . \sin\theta + \sin 2\theta . \cos\theta}{2\cos 2\theta \cos\theta + \sin 2\theta \cdot \sin\theta}$$

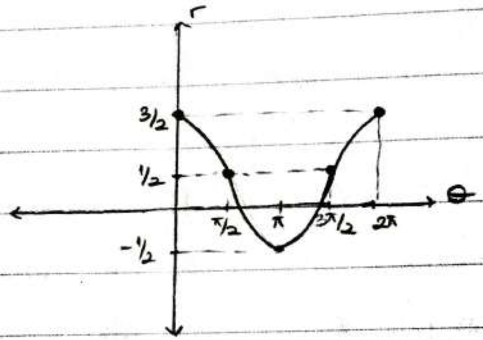
$$\left.\frac{dy}{dx}\right|_{\theta = \pi/4} = \frac{0 + \sqrt{2}/2}{0 - \sqrt{2}/2} = -1$$

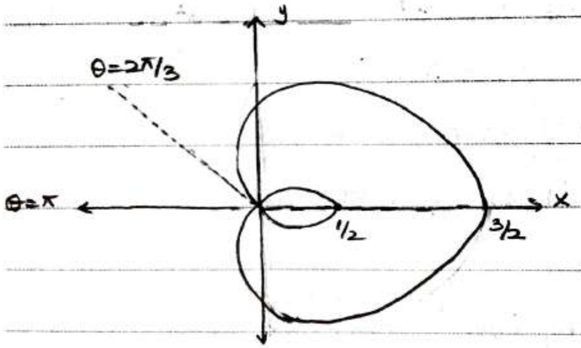
book 211

ex: Graph $r = \frac{1}{2} + \cos\theta$

$\cos(-\theta) = \cos\theta$

If (r, θ) is on the graph then $(r, -\theta)$ is also on the graph $\Rightarrow$ symmetry with respect to x-axis

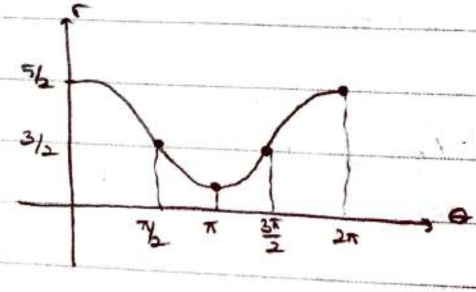




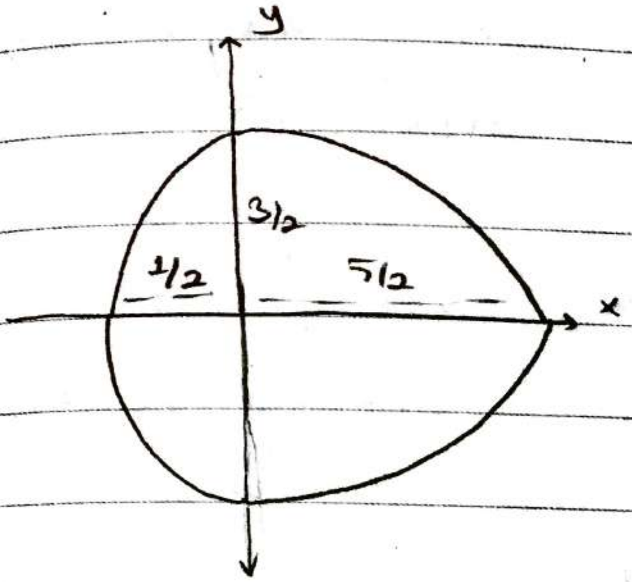
limaçon

ex: Graph the equation in the xy plane

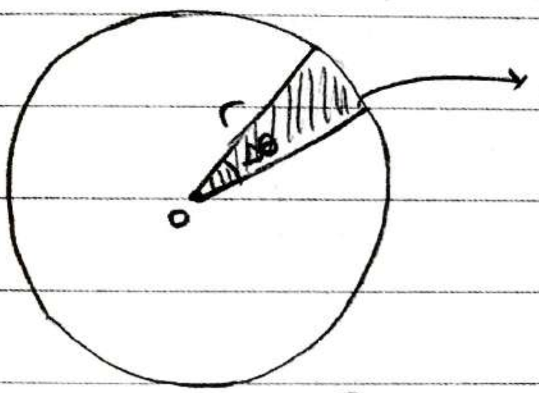
$r = \frac{3}{2} + \cos\theta$



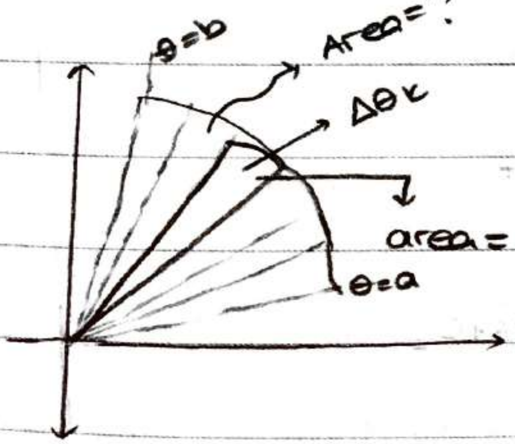
Graph is symetric x-axis



11.5 AREAS AND LENGTHS IN POLAR COORDINATES



$$\text{area} = \pi r^2 = \frac{\Delta\theta}{2\pi} = \frac{1}{2} r^2 \Delta\theta$$



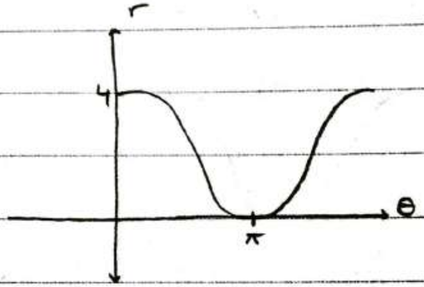
$$\text{area} = \frac{1}{2} r_k^2 \cdot \Delta\theta_k$$

$$\text{Area} \approx \sum_{k=1}^{n} \frac{1}{2} r_k^2 \Delta\theta_k$$

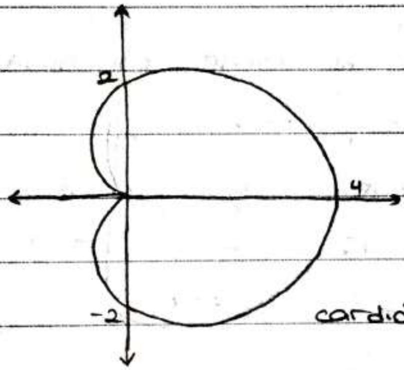
$n \to \infty$

$$\boxed{\int_{\theta=a}^{b} \frac{1}{2} r^2 \, d\theta = \text{Area}}$$

ex: Find the area of the region in the xy-plane enclosed by the cardioid $r=2(1+\cos\theta)$



symmetric x-axis



cardioid

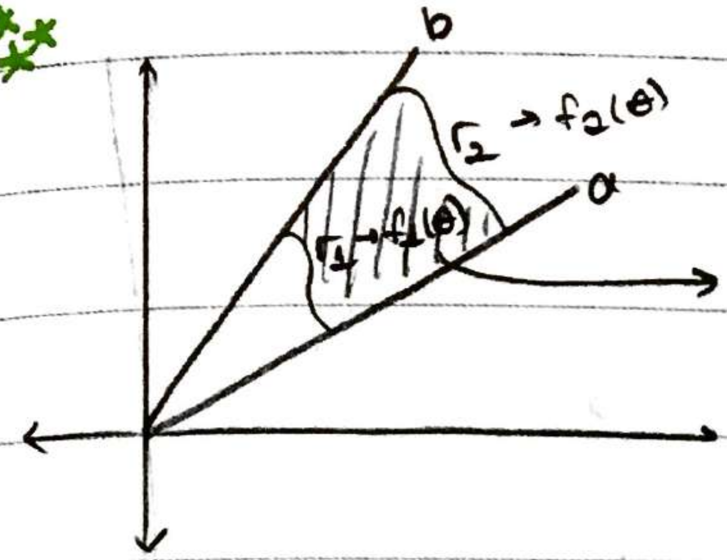
$$\text{Area} = 2\int_0^{\pi} \frac{1}{2}\cdot \underbrace{4\cdot(1+\cos\theta)^2}_{r^2}\, d\theta$$

$$= 4\int_0^{\pi} (1+2\cos\theta+\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}})\, d\theta$$

$$= 4\cdot\left(\theta+2\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\Big|_0^{\pi}$$

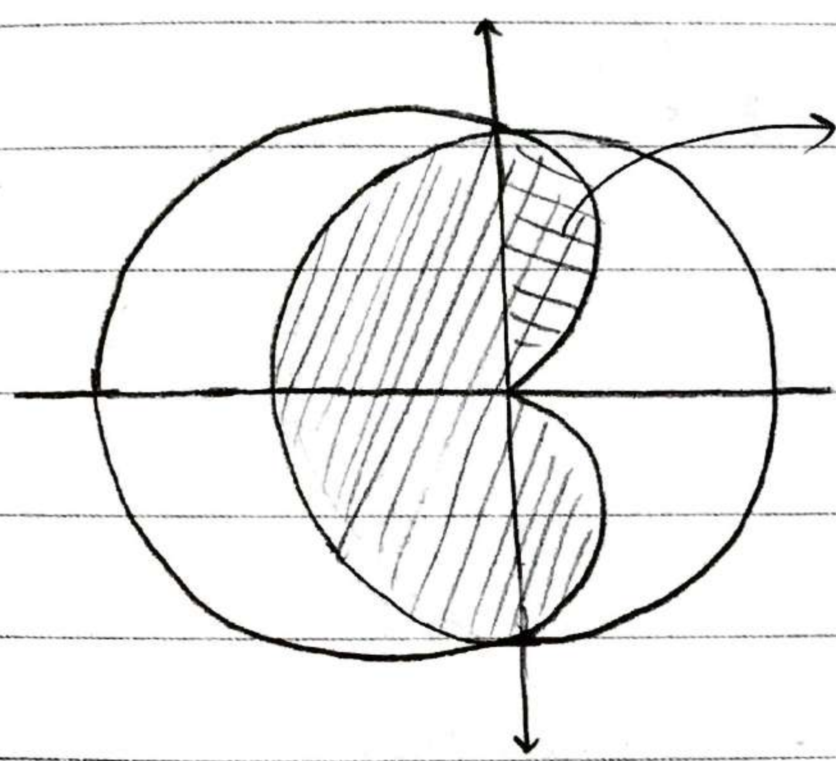
$$= 4\left[\left(\pi+0+\frac{\pi}{2}+0\right)-(0+0+0+0)\right]$$

$$= 6\pi$$



$$\text{Area} = \int_a^b (r_2^2 - r_1^2)\, d\theta$$

ex: Find the area of the region that lies inside the circle $r=1$ and the cardioid $r = 1-\cos\theta$



$$\text{Area} = \int_0^{\pi/2} \frac{1}{2}(1-\cos\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (1-2\cos\theta+\cos^2\theta)\, d\theta$$

$$= \left(\frac{\theta}{2} - \sin\theta + \frac{\theta}{4} + \frac{\sin 2\theta}{8}\right)\Big|_0^{\pi/2}$$

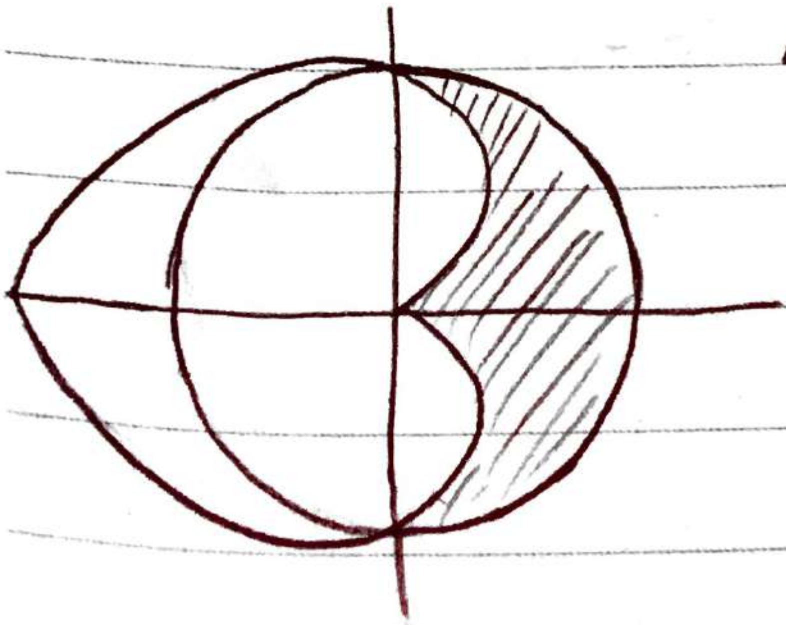
$$= \frac{3\pi}{8} - 1 + 0 - 0$$

$$= \frac{3\pi}{8} - 1$$

$$\text{Total Area} = 2\left(\frac{3\pi}{8} - 1\right) + \frac{\pi}{2}$$

Area inside the half circle.

ex: Same problem but inside the circle outside the cardioid.



$$\text{Area} = 2\int_0^{\pi/2} \frac{1}{2}\left(1^2 - (1-\cos\theta)^2\right) d\theta$$

$$= \int_0^{\pi/2} (1 - 1 + 2\cos\theta - \cos^2\theta)\, d\theta$$

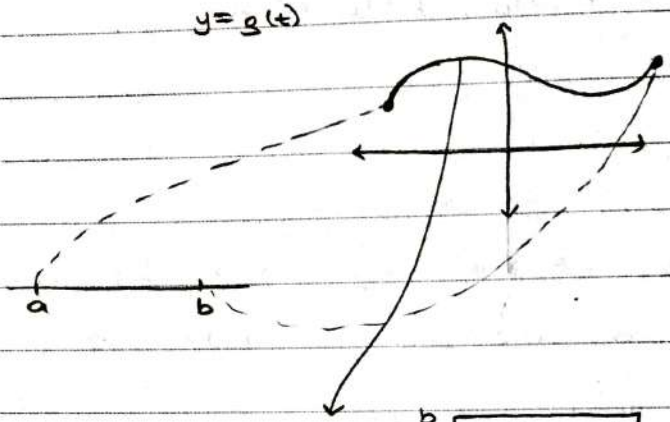
$$= 2\sin\theta - \frac{\sin 2\theta}{4} - \frac{\theta}{2}\Big|_0^{\pi/2}$$

$$= 2 - \pi/4$$

LENGTH OF A POLAR CURVE

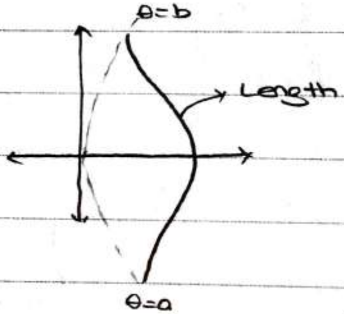
Recall $x = f(t)$

$y = g(t)$



$$\text{Length} = \int_{t=a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Polar curve $r = f(\theta)$, $a \leq \theta \leq b$



$x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$\left(f'(\theta)\cos\theta + f(\theta)(-\sin\theta)\right)^2 + \left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right)^2$$

$$= f'(\theta)^2\cos^2\theta + f(\theta)^2\sin^2\theta - f'(\theta)\cos\theta f(\theta)\sin\theta$$

$$+ f'(\theta)^2\sin^2\theta + f(\theta)^2\cos^2\theta - 2f'(\theta)\sin\theta f(\theta)\cos\theta$$

$$= f'(\theta)^2 - f(\theta)^2$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$